Pricing and matching under duopoly with imperfect buyer mobility

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Abstract

Recent contributions have explored how lack of buyer mobility affects pricing. For example, Burdett, Shi, and Wright (2001) envisage a two-stage game where, once prices are set by the firms, the buyers play a static subgame by choosing independently which firm to visit. We incorporate imperfect mobility in a duopolistic pricing game where the buyers are involved in a multi-stage game. The firms are shown to have an incentive to elicit loyalty on the part of the buyers by giving service priority to regular customers. Then equilibrium prices are higher than under a static buyer subgame; further, they converge to their value under perfect buyer mobility as the number of stages of the buyer subgame increases.

1 Introduction

Research on Bertrand-Edgeworth competition (price competition among capacity-constrained sellers) has tended to ignore the most obvious misallocations that would prevent maximization of consumers' and total surplus. More specifically, given the prices set by sellers of an identical good, at least the two following requirements for an efficient buyer allocation are usually assumed to hold: excess capacity at some firm cannot coexist with excess demand at other firms; relatively expensive firms receive no demand unless cheaper rivals are already producing at capacity. One possible, yet quite unrealistic, justification is to assume perfect mobility of buyers, that is, that any available capacity elsewhere is instantly detected and taken advantage of by any buyer who is rationed or asked to pay more at the chosen firm.

In contrast, in some recent models, after prices are set the buyers are playing a static game, each one selecting independently the firm to visit (see, among others, Peters, 1984 and 2000, Deneckere and Peck, 1995, Burdett, Shi, and Wright, 2001). This amounts to assuming no ex-post buyer mobility: if rationed at the selected firm, the buyer cannot move to other firms. The buyer's payoff thus depends on the probability of being served as well as the price at the chosen firm. The buyer allocation may be efficient only at a pure strategy equilibrium of the buyer game. Yet there are a multiplicity of such equilibria, all the more so the larger the number of buyers. Thus the attention has understandably been focused on the (symmetric) mixed strategy equilibrium, where misallocations occur with positive probability. Relying on this solution, the lack of buyer mobility proves to significantly affect equilibrium prices. Consider this simple setting, to be adopted throughout the paper. Two identical firms produce the same indivisible good at constant unit costs up to capacity. As in Burdett, Shi, and Wright (henceforth, BSW), each buyer demands inelastically one unit at any price not above the reservation price. Total capacity is fixed and equal to total demand. Under perfect mobility, both firms charging the reservation price is the unique equilibrium; in contrast, equilibrium prices are significantly less when the buyers are playing a static game. In fact, at equal prices expected output is less than each firm's capacity at the mixed strategy equilibrium of the buyer game. Consequently, with the rival charging the reservation price it pays to undercut since all buyers would then try the lower-priced firm.

Compared to the two aforementioned approaches, this paper intends to capture two features that are widely observed in real markets: goods are often purchased repeatedly over the time period for which prices are set; though buyers can move across the firms, mobility is too costly or unfeasible in a very short run, hence misallocations do occasionally occur. In our model, the buyers play a dynamic game of imperfect information once prices are set: at each stage each buyer selects one firm to visit without observing the choices made by the other buyers in the preceding stages.

To solve the buyer subgame we propose a variant of Kreps and Wilson's (1982) sequential equilibrium. Like sequential equilibrium, our "assessment equilibrium" involves a profile of strategies together with coherent beliefs at any information set where a buyer may be called upon to play. In a setting of repeat purchasing decisions the firms may give service priority to loyal customers rather than rationing purely at random among forthcoming buyers. It turns out that the firms have in fact a strong incentive to choose this "discriminatory" rationing rule. With such a rule in place, over a wide range of prices it is an assessment equilibrium for the buyers to obey a strategy of "conditional loyalty", prescribing loyalty if served by the previously chosen seller. Along the equilibrium path some efficient allocation

- where all buyers get served - is certainly achieved by the second stage of the buyer game. Most important, although this equilibrium may also exist when the firms ration randomly, conditional loyalty appears much more compelling under the discriminatory rule. Not only the benefits from conditional loyalty are then much higher and more easily perceived; also, unlike the random rule, the discriminatory rule disqualifies repeat playing of the mixed strategy equilibrium as an equilibrium of the dynamic buyer game.

The successful matching between buyers and sellers quickly obtaining at the assessment equilibrium of the dynamic buyer subgame has far-reaching implications on pricing. At a symmetric pure strategy equilibrium of the pricing game prices are higher than if buyers are involved in a static subgame; also, they converge to their value under perfect mobility as the number of stages of the buyer subgame goes to infinity. There is a clear intuition for this: each firm is going to quickly achieve full capacity utilization anyway, hence the incentives to undercutting the rival's price become negligible as the number of periods for which prices are set increases.

As already said, our model is similar in methodoloy to recent game-theoretic analyses of price determination under limited buyer mobility. However, in our analysis the buyers make repeat quantity and visiting decisions, which proves to have remarkable implications on pricing. It must also be noted that some of our results are similar to those obtained, through a different methodology, by Kirman and Vriend (2001).[1] These authors build an agent-based computational economics model upon the assumption of adaptive behaviour: among the actions taken in the past by the agents, those actions that gave better outcomes are more likely to be adopted in the future. One result of their model is the gradual emergence, among the buyers, of an attitude of loyalty to sellers previously visited and, among the sellers, of a favorable attitude towards repeat buyers.

The remainder of the paper is organized as follows. Section 2 considers a pricing game when the buyers are involved in a static game after prices are set by the duopolists. After reviewing the two-seller two-buyer case (already in BSW, along with more general ones), we turn to the case of any (even) number z of buyers facing the duopolists (whose total capacity is always assumed to equal total demand). We are able to derive the closed form solution for equilibrium prices with z as argument. Section 3 analyzes price setting when demand is made repeatedly by the buyers over $T+1$ stages:

[1] The difference between their approach and the game-theoretic approach also adopted in my earlier work on buyer loyalty (1996) is discussed thoroughly in Kirman and Vriend (2001).

we derive the closed form solution for equilibrium prices with z and T as arguments. Section 4 briefly concludes.

2 Pricing under one-period purchasing

2.1 The basic setting

Two firms, A and B, produce the same indivisible good, each with a given capacity $\overline{y}$. Any quantity up to $\overline{y}$ is obtained at constant unit costs (normalized to 0). There is a set $\mathcal{Z}=\{a, ..., h, ..., z\}$ of z identical buyers. Prices are set independently and simultaneously by the firms. Along with capacities, prices are known to the buyers who choose simultaneously and independently which firm to visit and how much to demand. Then each firm produces its capacity or its forthcoming demand, whichever is smaller. In this section, the buyers are playing a static game after the setting of prices. Every buyer demands inelastically one unit so long as the price does not exceed the reservation price, normalized to 1. Thus each firm chooses a price in the set $\mathcal{P} = [0, 1]$.

At any pair $(p_A, p_B) \in \mathcal{P}^2$ individual demand is equal to 1; granted this, buyer h's action space is simply denoted by $\{f_h\} = \{A, B\}$, where $f_h = A$ is the action of visiting firm A. A mixed strategy by buyer h is $\sigma_h = (\upsilon_h, 1 - \upsilon_h)$, where υ_h and $1 - \upsilon_h$ are the probabilities that h visits A and B, respectively. For brevity we henceforth write the buyer strategy just as υ_h and, accordingly, the space of mixed strategies as the unit interval, $I = [0, 1]$. We denote by $\pi(h_A^s)$ $(\pi(h_B^s))$ the probability of buyer h being served conditional on visiting A (resp., B). Prospective buyers at a firm have the same service probability.

Buyers are risk neutral, hence buyer h seeks to maximize his expected surplus: this is $(1 - p_A)\pi(h_A^s)$ if visiting A and $(1 - p_B)\pi(h_B^s)$ if visiting B. Total capacity is assumed to be equal to total demand:

$$2\overline{y} = z, \tag{1}$$

so we are forced to assuming an even number of buyers. A useful benchmark is the case of perfect mobility, where the buyers can instantly and costlessly move across the firms. Then the pair $(p_A = 1, p_B = 1)$ is the unique equilibrium: charging the reservation price is in fact strictly dominant for it allows the firm selling at capacity regardless of the rival's price.

2.2 The two-buyer case

We begin with the duopolists facing two buyers. (Apart from minor refinements, most of the results in this subsection are in BSW.) For a wide subset of $\mathcal{P}^2$, the buyer game has a symmetric mixed strategy equilibrium along with nonsymmetric pure strategy ones. Denote the buyers by h and k. Conditional service probabilities at A and B are, respectively, $\pi(h_A^s) = \frac{\upsilon_k}{2} + 1 - \upsilon_k$ and $\pi(h_B^s) = \upsilon_k + \frac{1-\upsilon_k}{2}$ for h and $\pi(k_A^s) = \frac{\upsilon_h}{2} + 1 - \upsilon_h$ and $\pi(k_B^s) = \upsilon_h + \frac{1-\upsilon_h}{2}$ for k. An equilibrium in strictly mixed strategies is symmetric, with υ such that $(1 - p_A)(\frac{\upsilon}{2} + 1 - \upsilon) = (1 - p_B)(\upsilon + \frac{1-\upsilon}{2})$. This yields

$$\upsilon = \upsilon(p_A, p_B) = \frac{1 - 2p_A + p_B}{2 - p_A - p_B}. \tag{2}$$

Thus a mixed strategy equilibrium (henceforth, a MSE) exists so long as

$$2p_A - 1 < p_B < \frac{1 + p_A}{2}. \tag{3}$$

Holding (3), two pure strategy equilibria (PSE) also exist, $(\upsilon_h = 1, \upsilon_k = 0)$ and $(\upsilon_h = 0, \upsilon_k = 1)$. At the MSE, each buyer has an expected surplus less than $\min\{1 - p_A, 1 - p_B\}$ and expected output is less than capacity for each firm. At each PSE, the buyers get $1 - p_A$ and $1 - p_B$ and the firms sell their capacity. Thus the PSEs Pareto-dominate the MSE. The buyers are assumed to take their decisions independently because of too high costs they should face to coordinate their actions.[2] Consequently, it is far from obvious that either PSE is played. In a sense, by allowing for misallocations of buyers the MSE seems to yield better predictions of the game outcome. Accordingly, holding (3) the buyers will be assumed to play the MSE.

At pairs of prices such that $2p_A - 1 > p_B$, the unique equilibrium is $(\upsilon_h = 0, \upsilon_k = 0)$; the equilibrium is likewise $(\upsilon_h = 1, \upsilon_k = 1)$ if $p_B > \frac{1+p_A}{2}$.[3] Special cases arise when $2p_A - 1 = p_B$ and when $p_B = \frac{1+p_A}{2}$. In the former, any strategy profile $(\upsilon_h = 0, 0 \leq \upsilon_k \leq 1)$ represents an equilibrium and so does any profile $(0 \leq \upsilon_h \leq 1, \upsilon_k = 0)$; yet it is reasonable to select equilibrium $(\upsilon_h = 0, \upsilon_k = 0)$ since $\upsilon_h = 0$ is weakly dominant.[4] By the same token, with $p_B = \frac{1+p_A}{2}$ one can select equilibrium $(\upsilon_h = 1, \upsilon_k = 1)$.

Turn now to pricing. Without loss of generality the analysis will henceforth be carried out in terms of firm A. Unlike under perfect mobility,

[2] This assumption is certainly most appropriate when there are many buyers.

[3] In either case, the equilibrium is ex-post inefficient. Let the equilibrium be $(\upsilon_h = 0, \upsilon_k = 0)$ and let h be rationed. If h could move to A, then he would get a positive surplus and benefit A without harming neither k nor B.

[4] With $2p_A - 1 = p_B$, it is only when $\upsilon_k = 0$ that $\upsilon_h = 1$ is also a best response.

($p_A = 1, p_B = 1$) is not an equilibrium. At equal prices $\upsilon = \frac{1}{2}$ at the MSE of the buyer game; hence expected output is $(\frac{1}{2})^2 + 2(\frac{1}{2})^2 = \frac{3}{4}$ for each firm. Consequently, with B charging the reservation price it pays A to slightly undercut, which raises A's expected profit from $\frac{3}{4}$ to almost 1 (both buyers would try A, where there is a chance of getting a tiny surplus). Denote by $E\Pi_A$ firm A's expected profits: $E\Pi_A = p_A E y_A$, where Ey_A is A's expected output. $dE\Pi_A/dp_A = \partial E\Pi_A/\partial p_A + (\partial E\Pi_A/\partial \upsilon)(\partial \upsilon/\partial p_A)$, that is,

$$\frac{dE\Pi_A}{dp_A} = Ey_A + p_A \frac{dEy_A}{d\upsilon}\frac{\partial \upsilon}{\partial p_A}. \tag{4}$$

Holding (3), $Ey_A = \upsilon^2 + 2\upsilon(1-\upsilon)$ with υ determined by (2), and $\partial \upsilon/\partial p_A = 3(p_B - 1)/(2 - p_A - p_B)^2$. Concavity of $E\Pi_A$ in p_A is readily established for the two-buyer case. In (4), Ey_A decreases as p_A increases (and υ correspondingly decreases). The term $p_A(dEy_A/d\upsilon)(\partial \upsilon/\partial p_A)$ decreases too since the positive factors p_A and $dEy_A/d\upsilon$ both increase ($dEy_A/d\upsilon$ is decreasing in υ) while the negative factor decreases ($\partial^2 \upsilon/\partial p_A^2 < 0$).

At an interior maximum $dE\Pi_A/dp_A = 0$. Looking for a symmetric equilibrium we also put $p_A = p_B \equiv p$ and $\upsilon = \frac{1}{2}$, obtaining ($p_A = \frac{1}{2}, p_B = \frac{1}{2}$).

2.3 The z-buyer case

Here we take the duopolists as facing any (even) number of buyers.[5] The first step is to identify the region of $\mathcal{P}^2$ where a symmetric MSE of the buyer game exists. Let $S_h(\upsilon_a, ..., \upsilon_h, ..., \upsilon_z)$ - or, more concisely, $S_h(\upsilon_h, \upsilon_{-h})$ - be h's expected surplus at strategy profile $(\upsilon_h, \upsilon_{-h})$. We now see that a symmetric MSE exists in the same region of $\mathcal{P}^2$ where it does with $z = 2$.

Lemma 1 *(i) Holding (3), a symmetric MSE of the buyer game exists; (ii) failing (3), the buyer game has no equilibrium in strictly mixed strategies.*

Proof. (i) The buyer game is symmetric: $S_h(\upsilon_h, \upsilon_{-h}) = S_k(\upsilon_k, \upsilon_{-k})$ $\forall h, k \in \mathcal{Z}, \upsilon_h = \upsilon_k, \upsilon_{-h} = \upsilon_{-k}$. One can determine the set of h's best responses to his opponents playing the same strategy υ. Repeating this at any $\upsilon \in I$ yields a correspondence $\mathcal{R}_h : I \to I$. All the sufficient conditions of Kakutani's theorem are met: I is a compact and convex subset of

[5] BSW generalize along different lines. For the case of equally sized firms, each firm is assumed to have unit capacity and equilibrium prices are found for any number of firms and buyers. Thus, given the number of firms, total demand increases relative to total capacity as the number of buyers increases.

the (one-dimensional) Euclidean space, $\mathcal{R}_h$ is nonempty, convex, and upper hemicontinuous for all $\upsilon \in I$. Thus $\exists\ \upsilon : \upsilon \in \mathcal{R}_h(\upsilon)$. By symmetry, $\mathcal{R}_h$ is the same for all $h \in \mathcal{Z}$, hence $(\upsilon, ..., \upsilon)$ is an equilibrium. Consequently, a symmetric MSE must exist if a symmetric PSE does not. The symmetric pure strategy profile $(\upsilon_a = 1, ..., \upsilon_z = 1)$ is ruled out as an equilibrium if $1 - p_B > \frac{1-p_A}{2}$; similarly, $1 - p_A > \frac{1-p_B}{2}$ rules out $(\upsilon_a = 0, ..., \upsilon_z = 0)$. These inequalities together constitute (3).

(ii) Let $2p_A - 1 > p_B$. Then $\upsilon_h = 0$ is strictly dominant, disqualifying any strictly mixed strategy profile as an equilibrium. In the special case where $2p_A - 1 = p_B$, $\upsilon_h = 0$ is the unique best response to all $k \neq h$ playing a strictly mixed strategy: again a strictly mixed strategy profile is ruled out as an equilibrium. One can argue likewise when $p_B > \frac{1+p_A}{2}$ and when $p_B = \frac{1+p_A}{2}$. ■

We now characterize the symmetric MSE of the buyer game. With all $k \neq h$ playing υ, the number of them at a firm, l, is binomial, with probability distribution $\binom{z-1}{l} \upsilon^l (1-\upsilon)^{z-1-l}$ and $\binom{z-1}{l} (1-\upsilon)^l \upsilon^{z-1-l}$, respectively, for A and B. Denote by $[\pi(h_A^s)]_{\upsilon_k=\upsilon}$ and $[\pi(h_B^s)]_{\upsilon_k=\upsilon}$ buyer h's service probability conditional on visiting A and B, respectively, when all $k \neq h$ visit A with probability υ. We then have:

$$[\pi(h_A^s)]_{\upsilon_k=\upsilon} = \sum_{l=0}^{z-1} \binom{z-1}{l} \upsilon^l (1-\upsilon)^{z-1-l} \min\left(1, \frac{z/2}{l+1}\right) \tag{5}$$

and

$$[\pi(h_B^s)]_{\upsilon_k=\upsilon} = \sum_{l=0}^{z-1} \binom{z-1}{l} (1-\upsilon)^l \upsilon^{z-1-l} \min\left(1, \frac{z/2}{l+1}\right) = 0. \tag{6}$$

When υ is the symmetric equilibrium strategy, then h is indifferent between A or B. Denote by $\varphi(\upsilon, p_A, p_B) = 0$ the function implicitly relating υ to p_A and p_B at the symmetric MSE, that is,

$$\varphi(\upsilon, p_A, p_B) = (1-p_A)\,[\pi(h_A^s)]_{\upsilon_k=\upsilon} - (1-p_B)\,[\pi(h_B^s)]_{\upsilon_k=\upsilon} = 0. \tag{7}$$

Implicit differentiation of (7) yields:

$$\frac{\partial \upsilon}{\partial p_A} = -\frac{\partial \varphi / \partial p_A}{\partial \varphi / \partial \upsilon} = \frac{[\pi(h_A^s)]_{\upsilon_k=\upsilon}}{(1-p_A)\left[\frac{d\pi(h_A^s)}{d\upsilon}\right]_{\upsilon_k=\upsilon} - (1-p_B)\left[\frac{d\pi(h_B^s)}{d\upsilon}\right]_{\upsilon_k=\upsilon}}, \quad (8)$$

where

$$\left[\frac{d\pi(h_A^s)}{d\upsilon}\right]_{\upsilon_k=\upsilon} = \sum_{l=0}^{z-1} l \begin{pmatrix} z-1 \\ l \end{pmatrix} \upsilon^{l-1}(1-\upsilon)^{z-1-l} \min\left(1, \frac{z/2}{l+1}\right) - \sum_{l=0}^{z-1} (z-1-l) \begin{pmatrix} z-1 \\ l \end{pmatrix} \upsilon^{l}(1-\upsilon)^{z-2-l} \min\left(1, \frac{z/2}{l+1}\right) \quad (9)$$

and

$$\left[\frac{d\pi(h_B^s)}{d\upsilon}\right]_{\upsilon_k=\upsilon} = -\sum_{l=0}^{z-1} l \begin{pmatrix} z-1 \\ l \end{pmatrix} (1-\upsilon)^{l-1}\upsilon^{z-1-l} \min\left(1, \frac{z/2}{l+1}\right) + \sum_{l=0}^{z-1} (z-1-l) \begin{pmatrix} z-1 \\ l \end{pmatrix} (1-\upsilon)^{l}\upsilon^{z-2-l} \min\left(1, \frac{z/2}{l+1}\right). \quad (10)$$

For subsequent use we determine $\partial \upsilon / \partial p_A$ when $p_A = p_B \equiv p$:

$$\left[\frac{\partial \upsilon}{\partial p_A}\right]_{p_A=p_B\equiv p} = \frac{\sum_{l=0}^{z-1} \begin{pmatrix} z-1 \\ l \end{pmatrix} \left(\frac{1}{2}\right)^{z-1} \min\left(1, \frac{z/2}{l+1}\right)}{(1-p)\sum_{l=0}^{z-1} \begin{pmatrix} z-1 \\ l \end{pmatrix} \left(\frac{1}{2}\right)^{z-2} (4l-2z+2) \min\left(1, \frac{z/2}{l+1}\right)}. \quad (11)$$

A more concise notation would be:

$$\left[\frac{\partial \upsilon}{\partial p_A}\right]_{p_A=p_B\equiv p} = \frac{[\pi(h^s)]_{\upsilon_k=\frac{1}{2}}}{2(1-p)\left[d\pi(h_A^s)/d\upsilon\right]_{\upsilon_k=\upsilon=\frac{1}{2}}}, \quad (11')$$

where

$$[\pi(h^s)]_{\upsilon_k=\frac{1}{2}} \equiv \sum_{l=0}^{z-1} \begin{pmatrix} z-1 \\ l \end{pmatrix} \left(\frac{1}{2}\right)^{z-1} \min\left(1, \frac{z/2}{l+1}\right) \quad (12)$$

and

$$\left[\frac{d\pi(h_A^s)}{d\upsilon}\right]_{\upsilon_k=\upsilon=\frac{1}{2}} = \sum_{l=0}^{z-1} \binom{z-1}{l} \left(\frac{1}{2}\right)^{z-2} (2l-z+1)\min\left(1, \frac{z/2}{l+1}\right). \tag{13}$$

To clarify the new notation in (11'), note that $[\pi(h^s)]_{\upsilon_k=\frac{1}{2}}$ is in fact the probability of h being served at either firm when $\upsilon_k = \frac{1}{2}$ for any $k \neq h$; stated another way, it is the probability of any buyer being served at the symmetric MSE of the buyer game when $p_A = p_B$. Furthermore, look back at $\partial\varphi/\partial\upsilon$ (see (7) and (8)) and note that $[d\pi(h_A^s)/d\upsilon]_{\upsilon_k=\upsilon=\frac{1}{2}} = -\,[d\pi(h_B^s)/d\upsilon]_{\upsilon_k=\upsilon=\frac{1}{2}}$: then it is understood that, in (11), $(1-p)$ is in fact multiplied by $2[d\pi(h_A^s)/d\upsilon]_{\upsilon_k=\upsilon=\frac{1}{2}}$. For any $\upsilon \in (0,1)$, $[d\pi(h_A^s)/d\upsilon]_{\upsilon_k=\upsilon} < 0$ and $[d\pi(h_B^s)/d\upsilon]_{\upsilon_k=\upsilon} > 0$: when υ_k increases for all $k \neq h$, buyer h's service prospects deteriorate at A and improve at B.

Let $[Ey]_{\upsilon=\frac{1}{2}}$ be the firm's expected output when $\upsilon_h = \frac{1}{2}$ for any $h \in \mathcal{Z}$:

$$[Ey]_{\upsilon=\frac{1}{2}} = \sum_{l=0}^{z} \binom{z}{l} \left(\frac{1}{2}\right)^{z} \min\left(l, \frac{z}{2}\right). \tag{14}$$

Clearly,

$$[\pi(h^s)]_{\upsilon_k=\frac{1}{2}} = \frac{[Ey]_{\upsilon=\frac{1}{2}}}{\overline{y}} = \frac{[Ey]_{\upsilon=\frac{1}{2}}}{z/2}. \tag{15}$$

A few properties of the variables just introduced are now established.

Lemma 2 *(i)* $[\pi(h^s)]_{\upsilon_k=\frac{1}{2}}$ *increases in* z*, converging to 1 as* $z \to \infty$*; (ii)* $[d\pi(h_A^s)/d\upsilon]_{\upsilon_k=\upsilon=\frac{1}{2}}$ *decreases in* z*, converging to* -1 *as* $z \to \infty$*; (iii)* $[\partial\upsilon/\partial p_A]_{p_A=p_B\equiv p}$ *increases in* z*, converging to* $-1/2(1-p)$ *as* $z \to \infty$; *(iv)* $[\partial\upsilon/\partial p_A]_{p_A=p_B\equiv p}$ *decreases in* p*, converging to* $-\infty$ *as* $p \to 1$.

Proof. For (i), (ii), and (iii) see the Appendix; (iv) follows from (11'). ■

Remarks We are especially interested at the intuition for parts (i) and (iv). Let us begin with (i). With $p_A = p_B$, all buyers are served at any PSE of the buyer game. Hence, $1 - [\pi(h^s)]_{\upsilon_k=\frac{1}{2}}$ is the percapita loss in total surplus resulting from absence of buyer coordination (which prevents them from playing any of the several PSEs). That $\lim_{z\to\infty}[\pi(h^s)]_{\upsilon_k=\frac{1}{2}} = 1$ is clearly suggested by the second column in Table 1. We now see how this result can also be derived by the weak law of large numbers. Recall that, with $\upsilon_k = \frac{1}{2}$ for any $k \neq h$, the number of $k \neq h$ at either firm is binomial with mean $\frac{1}{2}(z-1)$. Hence, the fraction $\frac{l}{z-1}$ of $k \neq h$ visiting a firm has mean $\frac{1}{2}$. According to Bernoulli's theorem,

$$\lim_{z\to\infty} \Pr\left(\frac{1}{2} - \varepsilon \leq \frac{l}{z-1} \leq \frac{1}{2} + \varepsilon\right) = 1 \quad \forall \epsilon > 0. \tag{16}$$

A lower bound on $[\pi(h^s)]_{\upsilon_k=\frac{1}{2}}$ is found by noting that

$$[\pi(h^s)]_{\upsilon_k=\frac{1}{2}} > \Pr\left(\frac{l}{z-1} \leq \frac{1}{2} + \varepsilon\right) \frac{z}{2} \frac{1}{(z-1)(\frac{1}{2}+\varepsilon)+1} =$$
$$\Pr\left(\frac{l}{z-1} \leq \frac{1}{2} + \varepsilon\right) \frac{1}{1+2\varepsilon+\frac{1}{z}(1-2\varepsilon)}.$$

In view of (16), $\lim_{z\to\infty} \Pr\left(\frac{l}{z-1} \leq \frac{1}{2} + \varepsilon\right) = 1 \; \forall \epsilon > 0$; also, $\lim_{z\to\infty}[1/(1 + 2\varepsilon + \frac{1}{z}(1-2\varepsilon)] = \frac{1}{1+2\varepsilon}$, hence $\lim_{z\to\infty} [\pi(h^s)]_{\upsilon_k=\frac{1}{2}} = 1$.

Now we get the intuition of part (iv). Recall that (7) holds at the symmetric MSE of the buyer game. Starting from any pair $(p_A = p, p_B = p)$, a unilateral change $\Delta p_A < 0$ must result in a change $\Delta\upsilon > 0$ such that $(1 - p - \Delta p_A)\,[\pi(h_A^s)]_{\upsilon_k=\frac{1}{2}+\Delta\upsilon} = (1-p)\,[\pi(h_B^s)]_{\upsilon_k=\frac{1}{2}+\Delta\upsilon}$. It derives from this condition that $\Delta\upsilon$ increases with p, converging to $\frac{1}{2}$ (the probability of picking A converging to 1) as p converges to 1, so $\lim_{p\to 1} [\partial\upsilon/\partial p_A]_{p_A=p_B\equiv p} = -\infty$. □

We now address price determination. Holding (3), the symmetric MSE of the buyer game is played, hence (4) becomes

$$\left[\frac{dE\Pi_A}{dp_A}\right]_{\upsilon_h=\upsilon} = [Ey_A]_{\upsilon_h=\upsilon} + p_A \left[\frac{dEy_A}{d\upsilon}\right]_{\upsilon_h=\upsilon} \frac{\partial\upsilon}{\partial p_A}, \tag{4'}$$

where

$$[Ey_A]_{\upsilon_h=\upsilon} = \sum_{l=0}^{z} \binom{z}{l} \upsilon^l (1-\upsilon)^{z-l} \min\left(l, \frac{z}{2}\right), \tag{17}$$

with υ implicitly defined by (7), and

$$\left[\frac{dEy_A}{d\upsilon}\right]_{\upsilon_h=\upsilon} = \sum_{l=0}^{z} l \begin{pmatrix} z \\ l \end{pmatrix} \upsilon^{l-1}(1-\upsilon)^{z-l} \min\left(l, \frac{z}{2}\right)$$
$$-\sum_{l=0}^{z}(z-l) \begin{pmatrix} z \\ l \end{pmatrix} \upsilon^{l}(1-\upsilon)^{z-1-l} \min\left(l, \frac{z}{2}\right). \tag{18}$$

The two following results are important in that they imply that the maximization problem faced by the firm has a unique interior solution.

Lemma 3 *(i) For any* $p_B \in (0,1)$*, denote by* $p_A^*(p_B)$ *any* p_A *such that* $dE\Pi_A/dp_A = 0$*. Then* $p_A^*(p_B) \in \left(\max\{2p_B - 1, 0\}, \frac{1+p_B}{2}\right)$.

(ii) $E\Pi_A$ *is concave.*

Proof. (i) From inspection of (4') one can check that $[dE\Pi_A/dp_A]_{\upsilon_h=\upsilon}$ is continuous in p_A for any $p_A \in (0,1)$, $[dE\Pi_A/dp_A]_{\upsilon_h=\upsilon} > 0$ at $p_A = 0$, and $[dE\Pi_A/dp_A]_{\upsilon_h=\upsilon} < 0$ at any $p_A \in \left[\frac{1+p_B}{2}, 1\right]$ (where $\upsilon = 0$). Note that, if $p_B < \frac{1}{2}$, then $\upsilon < 1$ for any $p_A \in [0,1]$ while, with $p_B \geq \frac{1}{2}$, then $\upsilon = 1$ at any $p_A \in [0, 2p_B - 1]$ and $[dE\Pi_A/dp_A]_{\upsilon_h=\upsilon} = z/2$ at any such p_A. Therefore some $p_A^*(p_B))$ must exist and $p_A^*(p_B) \in \left(\max\{2p_B - 1, 0\}, \frac{1+p_B}{2}\right)$.

(ii) In the Appendix. ■

Corollary $p_A^*(p_B)$ *is unique and* $p_A^*(p_B) = \arg\max_{p_A} E\Pi_A(p_A)$.

We are now in a position to solving the pricing game.

Proposition 1 *(i) At the unique simmetric pure strategy equilibrium of the pricing game,* $(p_A = p^*, p_B = p^*)$*, where*

$$p^* = \left(1 - \frac{1}{2\left[d\pi(h_A^s)/d\upsilon\right]_{\upsilon_k=\upsilon=\frac{1}{2}}}\right)^{-1}; \tag{19}$$

(ii) $p^* \in \left[\frac{1}{2}, \frac{2}{3}\right)$ *and increases monotonically in* z.

Proof. (i) Looking for any symmetric equilibrium, let $p_A = p_B \equiv p$ and hence $\upsilon = \frac{1}{2}$. Next, insert (11') into (4'), to obtain

$$\left[\frac{dE\Pi_A}{dp_A}\right]_{p_A=p_B\equiv p} = [Ey_A]_{\upsilon=\frac{1}{2}} + p\left[\frac{dEy_A}{d\upsilon}\right]_{\upsilon=\frac{1}{2}} \frac{[\pi(h^s)]_{\upsilon_k=\frac{1}{2}}}{2(1-p)\left[d\pi(h_A^s)/d\upsilon\right]_{\upsilon_k=\upsilon=\frac{1}{2}}}, \tag{20}$$

To find $[dEy_A/dv]_{v=\frac{1}{2}}$ it should be noted that the demand l forthcoming to the firm when $v_h = \frac{1}{2}$ for any $h \in Z$ has probability distribution $\binom{z}{l}(1/2)^z$, mean $z/2$, and variance $z/4$. It follows that:[6]

$$\left[\frac{dEy_A}{dv}\right]_{v=\frac{1}{2}} = \frac{z}{2}. \tag{21}$$

Thus (20) becomes

$$\left[\frac{dE\Pi_A}{dp_A}\right]_{p_A=p_B\equiv p} = [Ey_A]_{v=\frac{1}{2}} + \frac{z}{2}\frac{p}{(1-p)}\frac{[\pi(h^s)]_{v_k=\frac{1}{2}}}{2\left[d\pi(h^s_A)/dv\right]_{v_k=v=\frac{1}{2}}}. \tag{20'}$$

The price at a symmetric pure strategy equilibrium is found by setting (20') equal to zero. Taking (15) into account, this leads to (19).

(ii) This follows from part (ii) of Lemma 2. ■

Remarks Thus, regardless of z, the equilibrium price is bounded away from the reservation price. There is a clear intuition for Proposition 1. At equilibrium, the marginal cost and marginal benefit of a unilateral price reduction are equal. At a symmetric equilibrium, the marginal cost per unit of capacity of lowering p_A is $[Ey_A]_{v=\frac{1}{2}}/(z/2)$ (A's output is now sold at a lower price): it converges to 1 as $z \to \infty$ (recall (15) and part (i) of Lemma 2). The marginal benefit per unit of capacity is $-p\,[dEy_A/dp_A]_{v=\frac{1}{2}}/(z/2)$ (A's expected output increases when p_A decreases), or $-p\,[dEy_A/dv]_{v=\frac{1}{2}}[\partial v/\partial p_A]_{p_A=p_B\equiv p}/(z/2) = -p\,[\partial v/\partial p_A]_{p_A=p_B\equiv p}$: by part (iv) of Lemma 2, this is increasing in p, going to infinity as $p \to 1$. In view of this, p^* increases in z converging to a limit lower than 1. Finally, equating marginal cost and marginal benefit in the limit (that is, putting $1 = -\frac{p}{2(1-p)}$) yields $\lim_{z\to\infty} p^* = \frac{2}{3}$. Convergence of p^* to $\frac{2}{3}$ is illustrated by the third column of Table 1. □

[6] Inserting $v = \frac{1}{2}$ into (18) leads to

$$\left[\frac{dEy_A}{dv}\right]_{v=\frac{1}{2}} = \sum_{l=0}^{z}\binom{z}{l}(2l-z)\left(\frac{1}{2}\right)^{z-1}\min\left(l,\frac{z}{2}\right) = 4\sum_{l=0}^{\frac{z}{2}-1}\binom{z}{l}\left(l-\frac{z}{2}\right)\left(\frac{1}{2}\right)^{z} l$$

$$+2z\sum_{l=0}^{\frac{z}{2}-1}\binom{z}{z-l}\left(z-l-\frac{z}{2}\right)\left(\frac{1}{2}\right)^{z} = 4\sum_{l=0}^{\frac{z}{2}-1}\binom{z}{l}\left(l-\frac{z}{2}\right)^2\left(\frac{1}{2}\right)^{z} = \frac{z}{2}.$$

3 Pricing under repeat purchasing

3.1 The buyer game

The buyers are now assumed to take repeat purchasing and visiting decisions, based on the pair of prices $(p_A, p_B) \in \mathcal{P}^2$ set by the firms at $t = 0$. Without loss of generality we take $p_A \geq p_B$. At each stage $t = 1, ..., T+1$ every buyer chooses which firm to visit and how much to demand, whereupon each firm produces the minimum between capacity and its forthcoming demand. This setting incorporates imperfect mobility in a simple way: if rationed, a buyer cannot switch to the other firm in the same stage.

The buyer does not observe the actions previously taken by the other buyers: we are envisaging a dynamic buyer game of imperfect information and simultaneous moves. For simplicity, the buyers are assumed to care only about their current payoff. Also, individual demand is one at each stage, no matter whether the buyer got served or rationed in the preceding stages.

In this setting, the firms might reward loyalty: rather than rationing forthcoming buyers at random, they might commit themselves to the following, discriminatory rule.

DEFINITION 1. According to **the discriminatory rationing rule**, when the firm receives more than $z/2$ buyers at t:

any forthcoming buyer is served with equal probability if $t = 1$; if $t > 1$, the firm serves any forthcoming buyer whom it served at $t-1$ and allocates randomly any remaining capacity among remaining forthcoming buyers. □

As a matter of fact, the firms often reward loyalty some way or another. Repeat purchasers may be offered better prices or higher-quality goods (Bulkley, 1992, Caminal and Matutes, 1990): examples are "frequent flyer" programs offered by airlines, discount coupons for the next purchase, and trading stamps at retailers (Crémer, 1984; Schumann, 1986; Banerjee and Summers, 1987; Klemperer, 1987). Alternatively, as assumed here, the firms might give service priority to loyal customers. As noted by Carlton and Perloff: "in many producer good industries, good customers often get the product during 'tight' times, and other customers must wait. [...] Such rationing has occurred in many industries, such as paper, chemicals, and metals" (1990, p. 522; see also Carlton, 1991, p. 253).

In what follows we first explore the implications of the discriminatory rationing rule, which will give us insights into the rationale for such a rule. In our context of fixed demands and capacities, the discriminatory rule guarantees future delivery to any currently satisfied buyer who keeps loyal. The implications are noteworthy. Let buyer h be served by firm B at some stage

t. Then loyalty is actually dominant for this buyer at $t+1$, for it guarantees getting the good at the lowest price. One immediate consequence is that repeat playing of the MSE of the static game cannot be an equilibrium of the dynamic buyer game. Furthermore, a buyer who gets rationed by B has no hope of being served by trying B again, if only the buyers currently served by B are subsequently taking their dominant action. Putting these two things together, it seems safe to predict that even boundedly rational buyers are going to be matched to sellers in a quite short time.

But we want to build a complete argument for fully rational buyers, showing that at an equilibrium of the dynamic buyer game the buyers abide by a norm of "conditional loyalty", one that prescribes keeping loyal if previously served. To pursue this task we need some more notation. Events and probabilities are now dated by a time index. At any stage t, we denote by $h_A^s(t)$ the event of buyer h being served if visiting A, by $z_A(t) = \#\{h : f_h(t) = A\}$ the number of buyers visiting A, and by $\widehat{z}_A(t) = \#\{k \neq h : f_k(t) = A\}$ the number of all such buyers but h (when $f_h(t) = A$). At any date $t \geq 2$ - just before stage t is played - buyer h is at an information set, denoted by $H(t)$, containing the buyer's experience thus far: $H(t)$ is a $(t-1)$-component vector, the τ-th component being an element of the set $\{h_A^s(\tau), h_A^r(\tau), h_B^s(\tau), h_B^r(\tau)\}$ for any $\tau = 1, ..., t-1$.

To solve the dynamic buyer game we develop a variant of Kreps and Wilson's (1987) sequential equilibrium, to be called "assessment equilibrium".[7] It is characterized as follows. At every information set where he may be called upon to move, the player has a belief on what has transpired, namely, a probability distribution over histories of the game thus far. An "assessment" is a profile of behavioural strategies along with a system of beliefs (one for any conceivable information set). Our assessment equilibrium is an assessment that meets basic consistency requirements, all of which featuring prominently in Kreps and Wilson. "Sequential rationality" extends to imperfect-information games the requirement that strategies be mutual best responses: at every information set each player's equilibrium strategy is an optimal response to other players obeying their equilibrium strategies from then on. Sequential rationality must hold at information sets on the equilibrium path - information sets that occur with positive probability when the players have always adhered to their equilibrium strategies - as well as at out-of-equilibrium information sets. Concerning coherence of beliefs with strategies, our assessment equilibrium imposes Kreps and Wilson's requirement of "structural consistency" rather than their more controversial

[7] Binmore (1992) defines so a weakened version of sequential equilibrium.

requirement of "consistency".[8] Structural consistency means that, in all contingencies, beliefs can be derived using Bayes' rule. More precisely, at information sets on the equilibrium path, beliefs are derived by Bayes' rule and the assumption that every other player has adhered to his equilibrium strategy thus far. At out-of-equilibrium information sets, beliefs are derived by Bayes' rule under some alternative assumption about the strategies the other players have played thus far. As illustrated later on, when dealing with such information sets the following restriction is conveniently placed upon beliefs, besides structural consistency.

ASSUMPTION 1. Suppose at some date buyer h is at an information set off the equilibrium path. Then h's belief allows for past deviations from their equilibrium strategy on the part of other buyers to the extent that this is needed to reconcile h's past experience with Bayes' rule. □

To shorten our argument, we rule out, by assumption, the most obvious mistake the buyers might do.

ASSUMPTION 2 No buyer ever plays a strictly dominated action. □

Of course, for a myopic buyer, playing a strictly dominated action - one entailing a lower expected payoff at that stage, regardless of the other buyers' current actions - is definitively wrong, no matter the future course of action. Every buyer should avoid making such an obvious mistake. Adherence to Assumption 2 involves a limitation in the analysis when looking for the equilibrium of the dynamic buyer game. Indeed, a complete action plan should also include prescriptions applying at information sets which might only arise after some buyer has played a strictly dominated action. Unfortunately, laying down the prescriptions of the equilibrium strategy applying at some of these information sets is not always easy. By Assumption 2 these difficulties are sidestepped by assigning zero probability to such information sets.

Incidentally, there are two circumstances under which a buyer has a strictly dominated action. With $2p_A - 1 > p_B$, visiting A at stage one yields a strictly lower expected payoff than visiting B, no matter what the other buyers are doing. Also, when $p_B < p_A$ switching to A at any stage $t \geq 2$ is a strictly dominated action for a buyer who got served by B at $t-1$.

Conditional loyalty is now incorporated into a strategy for the buyer game.

[8] For doubts about the latter, see Osborne and Rubistein (1994, pp. 224-225). Incidentally, though not included in the definition of sequential equilibrium, structural consistency was held by Kreps and Wilson to be implied by "consistency", a claim that was subsequently disproved by Kreps and Ramey (1987).

DEFINITION 2. **The strategy of conditional loyalty (henceforth, (Θ))** makes the following prescriptions to the buyer:

(a) With $2p_A - 1 < p_B$, at $t = 1$ play the equilibrium mixed strategy of the static buyer game; at any $t > 1$, keep loyal if served at $t - 1$ and switch between sellers if rationed;

(b) With $2p_A - 1 \geq p_B$, at $t = 1$ visit B with unit probability; at any $t \geq 2$, do as in (a). $\square$

The evolution of play when all buyers are obeying Θ is readily found.

Proposition 2 *If all buyers obey Θ, then each firm will have a stable body of $z/2$ customers at any $t \geq 2$.*

Proof. With all buyers obeying Θ, $z_A(2) = z_B(2) = z/2$ no matter the buyer allocation at $t = 1$. All buyers are thus certainly served at $t = 2$, hence they all keep loyal at $t = 3$, and so on. $\blacksquare$

Before making our case for conditional loyalty further notation must be introduced. Denote by ρ the allocation of all $k \neq h$ among the firms. At any date $t \geq 2$, we denote by $\mu(\rho(t-1) \mid H(t))$ buyer h's ex-post probability distribution over ρ in the stage just elapsed and by $\pi(\rho(t) \mid H(t))$ his ex-ante probability distribution over ρ for the incoming stage, both conditional on $H(t)$. From $H(t)$ buyer h can derive a belief, that is, an ex-post joint probability distribution over $\rho(\tau)$ at any $\tau = 1, .., t-1$. This allows h to compute $\mu(\widehat{z}_A(t-1) \mid H(t))$ and $\mu(\widehat{z}_B(t-1) \mid H(t))$, that is, an ex post probability distribution over the number of $k \neq h$ having visited A and B, respectively, at stage $t-1$. Next, assuming all $k \neq h$ are obeying Θ in the incoming stage t, buyer h can construct an ex-ante probability distribution over the number of $k \neq h$ visiting each firm at t, denoted by $\pi(\widehat{z}_A(t) \mid H(t))$ and $\pi(\widehat{z}_B(t) \mid H(t))$. Together with $\mu(\widehat{z}_A(t-1) \mid H(t))$ and $\mu(\widehat{z}_B(t-1) \mid H(t))$, this in turn allows h to estimate his own service probability at A and B, denoted by $\pi(h^s_A(t) \mid H(t))$ and $\pi(h^s_B(t) \mid H(t))$, respectively. For example, $\pi(h^s_A(3) \mid h^r_A(1), h^s_B(2))$ denotes the probability that buyer h is served if visiting A at $t = 3$, as assessed by h conditional on service history $H(3) = (h^r_A(1), h^s_B(2))$.

We now see that the buyers will be conditionally loyal at an equilibrium of the dynamic game.

Proposition 3 *Along with coherent beliefs, all buyers obeying Θ is an assessment equilibrium of the dynamic buyer game.*

Proof. Along the proof we will occasionally use Assumption 1, so it is worth illustrating it. Let $2p_A - 1 < p_B$ and concede validity of Proposition 3.

Suppose h's information set at date 3 is, say, $H(3) = (h^r_A(1), h^s_A(2))$. Then h is clearly off the equilibrium path at that date: h has deviated himself from Θ at stage two; furthermore, h infers from $H(3)$ that some buyer previously served by A has switched to B at $t = 2$, in violation of Θ. On the other hand, according to Assumption 1 any buyer who was served by B as well as any buyer who was rationed by A are believed to have obeyed Θ at stage two: indeed, as one can verify, believing so is consistent with $H(3)$.

Along the proof it is helpful to distinguish among stage one, stage two, and any subsequent stage.

Optimality of Θ *at* $t = 1$. Obeying Θ is by definition a mutual best response at $t = 1$.

Optimality of Θ *at* $t = 2$. At $t = 2$ obeying Θ is dominant when $h^s_B(1)$ or when $h^s_A(1)$ and $p_A = p_B$. With $h^r_A(1)$ or $h^r_B(1)$, switching between sellers is clearly h's best response to the other buyers playing Θ at $t + 1$.

So we are left with the case in which $H(2) = h^s_A(1)$ and $p_A > p_B$. Note that, if it were $2p_A - 1 > p_B$, then h would have played a strictly dominated action at stage one by visiting A. Therefore, by Assumption 2 we can restrict ourselves to the case $2p_A - 1 \leq p_B$. The case $2p_A - 1 = p_B$ is readily dealt with. All $k \neq h$ are believed to have obeyed Θ at stage one: consequently, $\mu(\widehat{z}_B(1) = z - 1 \mid h^s_A(1)) = 1$, implying $\pi(h^s_B(2) \mid h^s_A(1)) = 0$. More elaboration is needed if $2p_A - 1 < p_B$. Again the event $h^s_A(1)$ is consistent with all $k \neq h$ having obeyed Θ at stage one, that is, with every k having picked either firm with positive probability. Then h perceives to have a positive service probability if switching to B at stage two: there is in fact a chance of being served if $\widehat{z}_A(1) \geq z/2$, in which case, according to Θ, unsatisfied buyers are moving to B at stage two. While keeping loyal to A yields a surplus of $1 - p_A$, switching to B results in an expected surplus of $(1 - p_B)\pi(h^s_B(2) \mid h^s_A(1))$. So it has to be shown that

$$1 - p_A > (1 - p_B)\pi(h^s_B(2) \mid h^s_A(1)). \tag{22}$$

Note that $(1 - p_A)\pi(h^s_A(1)) = (1 - p_B)\pi(h^s_B(1))$: since all $k \neq h$ are held to have obeyed Θ at stage one, h was indifferent between A and B at that stage. Hence we would be done by showing that

$$\pi(h^s_B(2) \mid h^s_A(1)) < \pi(h^s_B(1)). \tag{23}$$

Note that

$$\pi(h^s_B(1)) = \pi\left(\widehat{z}_B(1) < z/2\right) + \sum_{l=z/2}^{z-1} \pi(\widehat{z}_B(1) = l)\frac{z/2}{l+1}, \tag{24}$$

where $\pi(\widehat{z}_B(1)=l)=\begin{pmatrix} z-1 \\ l \end{pmatrix}(1-v)^l v^{z-1-l}$. On the other hand,

$$\pi(h_B^s(2) \mid h_A^s(1)) = \sum_{l=0}^{z/2-1} \mu(\widehat{z}_B(1)=l \mid h_A^s(1)) \frac{(z/2)-l}{(z/2)-l+1}. \qquad (25)$$

This equation is readily understood. By moving to B at $t=2$, h has a chance of being served if B was faced with $l<z/2$ buyers at $t=1$. Then h would compete at B with $(z/2-l)$ buyers - those previously rationed by A, who are now moving to B - over an output of $z/2-l$. To see that the RHS of (25) is less than that of (24) it suffices to show that $\sum_{l=0}^{z/2-1}\mu(\widehat{z}_B(1)=l \mid h_A^s(1)) < \pi(\widehat{z}_B(1)<z/2)$. It must preliminarily be noted that $\pi(\widehat{z}_B(1)<z/2) = \pi(\widehat{z}_B(1)<z/2 \mid f_h(1)=A) = \pi(\widehat{z}_B(1)<z/2 \mid f_h(1)=B)$.[9] Consequently,

$$\begin{aligned}\pi(\widehat{z}_B(1)<z/2) &= \pi\left(\widehat{z}_B(1)<z/2, h_A^s(1)\right)+\pi(\widehat{z}_B(1)<z/2, h_A^r(1)) \\ &= \pi\left(h_A^s(1)\right)\mu(\widehat{z}_B(1)<z/2 \mid h_A^s(1))+\pi(h_A^r(1))\mu(\widehat{z}_B(1)<z/2 \mid h_A^r(1)) \\ &= \pi\left(h_A^s(1)\right)\mu(\widehat{z}_B(1)<z/2 \mid h_A^s(1))+1-\pi(h_A^s(1)). \end{aligned} \qquad (26)$$

The scrutiny of (26) reveals that $\mu\left(\widehat{z}_B(1)<z/2 \mid h_A^s(1)\right) < \pi(\widehat{z}_B(1)<z/2)$.

Optimality of Θ *at* $t \geq 2$. We begin supposing h at date t is at an information set on the equilibrium path. This means that h has obeyed Θ thus far and, by Proposition 2, that h has been served at $\tau=2,...,t-1$. Then obeying Θ at stage t results in a unit service probability while switching between firms results in a zero service probability.

Suppose next h at some date $t>2$ is at an information set off the equilibrium path. The argument follows the previous lines when $h_B^r(t-1)$ or $h_A^r(t-1)$ as well as when $h_B^s(t-1)$ or, with $p_A=p_B$, $h_A^s(t-1)$. So we are again left with the case in which $h_A^s(t-1)$ and $p_A>p_B$. This collection of information sets can be partitioned into the following subsets:

(a) $H(t)=(..., h_A^s(t-2), h_A^s(t-1))$. Note that for any such $H(t)$ to be off the equilibrium path the same must be so as for $H(t-1)=(..., h_A^s(t-2))$. By Assumption 1, at date t all $k \neq h$ are then believed to have obeyed Θ at stage $t-1$. On reflection, this implies $\mu(\widehat{z}_B(t-1)=z/2 \mid H(t))=1$ so that $\pi\left(h_B^s(t) \mid H(t)\right)=0$.

(b) $H(t)=(..., h_A^r(t-2), h_A^s(t-1))$. This reveals that at $t-1$ some buyer previously served by A has moved to B. Along with Assumption 1 this implies that $\mu(\widehat{z}_B(t-1) \geq z/2 \mid H(t))=1$ so that $\pi\left(h_B^s(t) \mid H(t)\right)=0$.

[9] Obviously, the probability that $\widehat{z}_B(1)<z/2$ does not depend on h's action at $t=1$.

(c) $H(t) = (..., h_B^s(t-2), h_A^s(t-1))$. By Assumption 2, we can limit ourselves to the case in which $p_A = p_B$. Optimality of Θ at t is then obvious.

(d) $H(t) = (..., h_B^r(t-2), h_A^s(t-1))$. This is consistent with all $k \neq h$ having obeyed Θ at $t-1$. Accordingly $\mu\left(\widehat{z}_B(t-1) = z/2 \mid H(t)\right) = 1$ and $\pi\left(h_B^s(t) \mid H(t)\right) = 0$. ■

Remarks It should be clear the type of learning that is taking place along the equilibrium path. Some efficient allocation (any such that $z_A(t) = z_B(t) = z/2$) is certainly achieved by $t = 2$ without buyer h actually knowing which firm any $k \neq h$ is going to visit in the incoming stage. The action currently made by any k depends on whether k was served at $t-1$, something which h can neither observe nor infer for sure (if $z > 2$). Yet h is able to predict the custom sizes at the two firms. For example, let $h_A^s(t-1)$. Then h predicts $\widehat{z}_A(t) = z/2 - 1$, which is correct if all $k \neq h$ are obeying Θ at t.

It is worth emphasizing that the market becomes segmented while the buyers are learning about each firm's custom size. Assume $p_A = p_B \equiv p$. Then at any $t \geq 2$ every buyer gets surplus $1-p$ on the equilibrium path. Yet the firms are ex post no longer equivalent to the buyers: at any $t \geq 2$ switching between sellers would prejudice the buyer's service prospects. □

3.2 Solving the entire game

At $t = 0$ the firms set prices whereupon the assessment equilibrium of the buyer subgame is played. Each firm is concerned with its (undiscounted) expected profits over the $T+1$ stages of the buyer game. This is written $\sum_{t=1}^{T+1} E\Pi_A(t) = E\Pi_A(1) + \sum_{t=2}^{T+1} E\Pi_A(t)$ for firm A. Looking for a symmetric pure strategy equilibrium of the pricing game leads to:

Proposition 4 *(i) At the unique symmetric pure strategy equilibrium ($p_A = p^{**}, p_B = p^{**}$), where*

$$p^{**} = \left[1 - \frac{1}{2\left[d\pi(h_A^s)/dv\right]_{v_k=v=\frac{1}{2}}\left(1 + T/\left[\pi(h^s)\right]_{v_k=v=\frac{1}{2}}\right)}\right]^{-1}; \qquad (27)$$

(ii) $p^{**} > p^*$, and p^{**} increases in T with $p^{**} \to 1$ as $T \to \infty$; (iii) p^{**} increases in z with $p^{**} \to \frac{2+2T}{3+2T}$ as $z \to \infty$.

Proof. (i) At any pair of prices meeting (3), $\sum_{t=1}^{T+1} E\Pi_A(t) = p_A\left[Ey_A\right]_{v_h=v} + p_A\frac{z}{2}T$ at the resulting assessment equilibrium of the dynamic buyer subgame.

We put $p_A = p_B \equiv p^{**}$ and $\upsilon = \frac{1}{2}$ into the FOC for an interior maximum. Recalling (21) and (11'), it is obtained:

$$[Ey_A]_{\upsilon=\frac{1}{2}} + \frac{z}{2}\frac{p^{**}}{(1-p^{**})}\frac{[\pi(h^s)]_{\upsilon_k=\frac{1}{2}}}{2\left[d\pi(h^s_A)/d\upsilon\right]_{\upsilon_k=\upsilon=\frac{1}{2}}} = -\frac{z}{2}T. \tag{28}$$

Solving (28) and using (15) leads to (27). That p^{**} is actually a best response to firm B charging p^{**} follows from two facts. First, concavity of $E\Pi_A(1)$ implies concavity of $\sum_{t=1}^{T+1} E\Pi_A(t)$ over the interval $\left[0, \frac{1+p_B}{2}\right]$ of p_A; consequently $\sum_{t=1}^{T+1} E\Pi_A(t)$ would be lower at any $p_A \neq p^{**}$ in this interval. Second, $d\sum_{t=1}^{T+1} E\Pi_A(t)/dp_A = \frac{z}{2}T$ for $p_A \in (\frac{1+p_B}{2}, 1]$ (where no demand is addressed to A at $t = 1$). Clearly, for p_A in this interval, the best option would be to set $p_A = 1$. This affords a total profits of $\frac{z}{2}T$ to A, which, as one can check, is less than $p^{**}\frac{z}{2}([\pi(h^s)]_{\upsilon_k=\frac{1}{2}} + T)$.

(ii) By comparing (27) with (19) it is seen that $p^{**} > p^*$. Also, p^{**} increases in T, converging to 1 as $T \to \infty$.

(iii) That p^{**} increases in z follows from part (iii) of Lemma 2 (recall (11')). Further, it follows from parts (i) and (ii) of Lemma 2 that $\lim_{z\to\infty} p^{**} = \frac{2+2T}{3+2T}$. ■

Remarks The intuition of part (ii) goes as follows. At any $(p_A = p, p_B = p)$ the marginal benefit of lowering p_A is $-p[dEy_A/dp_A]_{\upsilon=\frac{1}{2}}$ (A's expected output increases at $t = 1$). This is proportional to $\frac{p}{1-p}$, hence increasing in p and becoming indefinitely large as $p \to 1$. The marginal cost is now $\frac{z}{2}\left([\pi(h^s)]_{\upsilon_k=\frac{1}{2}} + T\right)$, the term $\frac{z}{2}T$ reflecting the fall in revenues at any $t \geq 2$. The marginal cost thus increases in T and becomes indefinitely large as $T \to \infty$. It follows from all this that $p^{**} > p^*$; also, p^{**} increases in T with $\lim_{T\to\infty} p^{**} = 1$. Thus the impact on equilibrium prices of imperfect mobility becomes less and less important as the number of stages of the buyer game increases: equilibrium prices under imperfect mobility converge to their value under perfect mobility.

An illustration of parts (ii) and (iii) is provided by the right part of Table 1, where p^{**} has been computed for different values of z and T.

3.3 More on the rationing rule

Adoption of the discriminatory rationing rule has thus far been taken for granted. The question naturally arising is whether the firms would act so in the first place. To begin, take the firms to be somehow committed to ration randomly among their forthcoming buyers. Then, it is immediately seen that, unlike under the discriminatory rule, with (p_A, p_B) meeting (3), repeat playing of the symmetric MSE of the static buyer game is an equilibrium of the dynamic game.

Existence of such an equilibrium clearly suggests that there is no guarantee that misallocations will disappear when the firms ration randomly. Yet, and somewhat surprisingly, this is still a possibility: conditional loyalty can be an assessment equilibrium.[10] Suppose buyer h is rationed at t : one can readily verify that, as long as all $k \neq h$ are obeying Θ at $t+1$, buyer h will be served with unit probability at $t+1$ if obeying Θ and with probability $z/(z+2)$ if deviating from Θ. Exactly the same service probabilities obtain if h is served at t. The implication is straightforward: even under random rationing, if prices are equal or sufficiently close to each other, then obeying Θ is an assessment equilibrium of the dynamic buyer game. The loss in service probability from unilaterally deviating from Θ equals $(1-z/(z+2))$. It deserves to be noted, though, that this loss is much less than under the discriminatory rule (when it equals 1 at any $t > 2$) and becoming smaller and smaller as z increases. Furthermore, less-than fully rational buyers may fail to recognize the benefits from conditional loyalty: unlike under the discriminatory rule, for a buyer who got served by the lower priced firm keeping loyal is not a dominant action. To keep the argument most simple, let us assume that repeat playing of the MSE of the static game is the equilibrium actually played by the buyers under random rationing. Then we can draw on Section 2 to solve the pricing game: with the random rule in place, the firms set prices at p^* at the (symmetric) pure-strategy equilibrium.

We now see what happens when both firms turn exogeneously from the random to the discriminatory rationing rule. The market becomes more efficient due to the full exploitation of capacity at $t \geq 2$: total surplus rises by $zT(1-[\pi(h^s)]_{\upsilon_k=\frac{1}{2}})$. The firms benefit from both the increased efficiency and their increased market power ($p^{**} > p^*$), whereas the buyers are worse off. The buyer is clearly harmed at $t = 1$, where his expected surplus falls by $(p^{**} - p^*)\,[\pi(h^s)]_{\upsilon_k=\frac{1}{2}}$. He is also worse off at any $t > 1$, though being

[10] This result can actually be extended to the case of any number of firms. In two earlier works we have argued that, with the firms charging the same exogenously given price, conditional loyalty is an assessment equilibrium of the dynamic buyer game with either rationing rule (De Francesco, 1996 and 1998).

now served for sure: $1 - p^{**} < (1 - p^{*}) \left[\pi(h^s)\right]_{v_k = \frac{1}{2}}$.[11]

At long last, we are able to endogenize the rationing rule. Let the firms have two simultaneous choice variables at $t = 0$: besides setting prices, they commit independently to either the random or the discriminatory rationing rule. Then there are two candidate symmetric equilibria: one in which the firms ration randomly and set prices at p^* and another where they adopt the discriminatory rule and charge p^{**}. The former is ruled out, though, because it pays the firm to unilaterally deviate to the discriminatory rule (even keeping its price unchanged). Note that loyalty is dominant for any buyer who gets served by the deviating firm. Consequently, repeat playing of the MSE of the static buyer game is no longer an equilibrium of the dynamic buyer subgame, while the assessment equilibrium in which the buyers obey Θ becomes intuitively compelling. Thus the deviation is expected to result in higher output and profits.

It can similarly be shown that it is an equilibrium for the firms to adopt the discriminatory rationing rule and set p^{**}. Consider a unilateral deviation to random rationing. Since one firm is still adopting the discriminatory rule, repeat playing of the MSE of the static buyer game cannot be an equilibrium, while the assessment equilibrium where the buyers obey Θ still retain its intuitive appeal. So the deviation under discussion does not affect expected profits of either firm.

4 Conclusion

We have examined Bertrand-Edgeworth competition for a symmetric duopoly in a setting where total capacity equals (an inelastic) total demand, the good is purchased repeatedly once prices are set, and the buyers are imperfectly mobile across firms. A strong case case has emerged for the firms to serve loyal customers first. Then being loyal if previously served is readily recognized by the buyers as the right thing to do. This leads to some efficient buyer allocation to be quickly reached and mantained forever after. The

[11] In view of (19) and (27), this condition amounts to

$$[\pi(h^s)]_{v_k=\frac{1}{2}} > \frac{1 - 2(1-T)[\frac{d\pi(h_A^s)}{dv}]_{v_k=v=\frac{1}{2}}}{1 - 2[\frac{d\pi(h_A^s)}{dv}]_{v_k=v=\frac{1}{2}}}.$$

This inequality is always met. In fact, taking account of part (ii) of Lemma 2, the LHS is maximal at $z = 2$ and $T = 2$. The maximum is zero because, as one can check from (9), $[\frac{d\pi(h_A^s)}{dv}]_{v_k=v=\frac{1}{2}} = -1/2$ at $z = 2$.

implications for pricing are straightforward. The gain from price undercutting becomes a short-lived one because the buyers will soon be perfectly matched to sellers anyway. As a result, equilibrium prices are higher than if the buyers were involved in a static buyer game; they actually converge to their value under perfect mobility as the number of stages of the buyer game increases.

While market efficiency improves when loyalty is rewarded, it is only the sellers who reap the benefits; the increase in their market power is large enough so as to make the buyers worse off. It would be interesting to check how this conclusion depends on the short-run setting of the present model. This is a task we leave to future research, which might analyze price competition with imperfect mobility in a long-run framework in which the number and the capacity of firms are endogenous.

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APPENDIX

Proof of Lemma 2. (i) In view of (14), (15) can be written

$$[\pi(h^s)]_{\upsilon_k=\frac{1}{2}} = \frac{2}{z}\sum_{l=0}^{\frac{z}{2}} \begin{pmatrix} z \\ l \end{pmatrix} \left(\frac{1}{2}\right)^z l + \sum_{l=\frac{z}{2}+1}^{z} \begin{pmatrix} z \\ l \end{pmatrix} \left(\frac{1}{2}\right)^z, \tag{29}$$

where l is binomial, with mean $z/2$ and standard deviation $\sqrt{z}/2$. It follows from symmetry and unimodality that

$$\sum_{l=\frac{z}{2}+1}^{z} \binom{z}{l} \left(\frac{1}{2}\right)^{z} = \frac{1}{2} - \frac{1}{2} \binom{z}{\frac{z}{2}} \left(\frac{1}{2}\right)^{z}. \tag{30}$$

Using the Stirling formula, $n! \approx \sqrt{2\pi} n^{(n+1/2)} e^{-n}$, it is obtained

$$\frac{1}{2} \binom{z}{z/2} \left(\frac{1}{2}\right)^{z} = \frac{1}{\sqrt{2}\sqrt{\pi}\sqrt{z}}. \tag{31}$$

On reflection, the mean of l can be written

$$\frac{z}{2} = \sum_{l=1}^{\frac{z}{2}} \binom{z}{l} \left(\frac{1}{2}\right)^{z} l + \sum_{s=0}^{\frac{z}{2}-1} \binom{z}{s} \left(\frac{1}{2}\right)^{z} (z-s). \tag{32}$$

Note that $\binom{z}{l} l = \binom{z}{s} (z-s)$ for any $l = 1, ..., \frac{z}{2}$, $s = l-1$, hence the two sums on the RHS of (32) are equal. Consequently,

$$\sum_{l=0}^{\frac{z}{2}} \binom{z}{l} \left(\frac{1}{2}\right)^{z} l = \frac{z}{4}. \tag{33}$$

Inserting (30), (31), and (33) into (29) yields

$$[\pi(h^s)]_{\upsilon_k=\frac{1}{2}} = 1 - \frac{1}{\sqrt{2}\sqrt{\pi}\sqrt{z}}. \tag{34}$$

Thus $[\pi(h^s)]_{\upsilon_k=\frac{1}{2}}$ increases in z with $\lim_{z\to\infty} [\pi(h^s)]_{\upsilon_k=\frac{1}{2}} = 1$.

(ii) Equation (13) can be written

$$\left[\frac{d\pi(h_A^s)}{d\upsilon}\right]_{\upsilon_k=\upsilon=\frac{1}{2}} = 4\sum_{l=0}^{z-1} \binom{z-1}{l} \left(\frac{1}{2}\right)^{z-1} (l+1) \min\left(1, \frac{z/2}{l+1}\right)$$
$$-2(z+1) \sum_{l=0}^{z-1} \binom{z-1}{l} \left(\frac{1}{2}\right)^{z-1} \min\left(1, \frac{z/2}{l+1}\right), \tag{13'}$$

or, more concisely, as

$$\left[\frac{d\pi(h_A^s)}{d\upsilon}\right]_{\upsilon_k=\upsilon=\frac{1}{2}} = 4\sum_{l=0}^{z-1} \binom{z-1}{l} \left(\frac{1}{2}\right)^{z-1} (l+1) \min\left(1, \frac{z/2}{l+1}\right)$$
$$-2(z+1) [\pi(h^s)]_{\upsilon_k=\frac{1}{2}}. \tag{13''}$$

Note that

$$4\sum_{l=0}^{z-1}\left(\begin{array}{c}z-1\\l\end{array}\right)\left(\frac{1}{2}\right)^{z-1}(l+1)\min\left(1,\frac{z/2}{l+1}\right)=$$
$$4\sum_{l=0}^{\frac{z}{2}-1}\left(\begin{array}{c}z-1\\l\end{array}\right)\left(\frac{1}{2}\right)^{z-1}l+4\sum_{l=0}^{\frac{z}{2}-1}\left(\begin{array}{c}z-1\\l\end{array}\right)\left(\frac{1}{2}\right)^{z-1}+$$
$$2z\sum_{l=\frac{z}{2}}^{z-1}\left(\begin{array}{c}z-1\\l\end{array}\right)\left(\frac{1}{2}\right)^{z-1}=4\sum_{l=0}^{\frac{z}{2}-1}\left(\begin{array}{c}z-1\\l\end{array}\right)\left(\frac{1}{2}\right)^{z-1}l+2+z. \quad (35)$$

Binomial l in $\sum_{l=0}^{\frac{z}{2}-1}\left(\begin{array}{c}z-1\\l\end{array}\right)(\frac{1}{2})^{z-1}l$ is symmetric and bimodal, with mean $\frac{z-1}{2}$ and standard deviation $\frac{\sqrt{z-1}}{2}$. The mean can be written as

$$\begin{aligned}\frac{z-1}{2} &= \sum_{l=1}^{\frac{z}{2}-1}\left(\begin{array}{c}z-1\\l\end{array}\right)\left(\frac{1}{2}\right)^{z-1}l+\left(\begin{array}{c}z-1\\z/2\end{array}\right)\left(\frac{1}{2}\right)^{z-1}\frac{z}{2}+\\ &\quad \sum_{s=0}^{\frac{z}{2}-2}\left(\begin{array}{c}z-1\\s\end{array}\right)\left(\frac{1}{2}\right)^{z-1}(z-1-s). \end{aligned} \quad (36)$$

The two sums on the RHS are equal because $\left(\begin{array}{c}z-1\\l\end{array}\right)l=\left(\begin{array}{c}z-1\\s\end{array}\right)(z-1-s)$ for any $l=1,...,\frac{z}{2}-1$ and $s=l-1$. Therefore,

$$\sum_{l=0}^{\frac{z}{2}-1}\left(\begin{array}{c}z-1\\l\end{array}\right)\left(\frac{1}{2}\right)^{z-1}l=\frac{z-1}{4}-\frac{1}{2}\left(\begin{array}{c}z-1\\\frac{z}{2}\end{array}\right)\left(\frac{1}{2}\right)^{z-1}\frac{z}{2}, \quad (37)$$

or, by applying the Stirling's formula,

$$\sum_{l=0}^{\frac{z}{2}-1}\left(\begin{array}{c}z-1\\l\end{array}\right)\left(\frac{1}{2}\right)^{z-1}l=\frac{z-1}{4}-\frac{\sqrt{2}\sqrt{z}}{4\sqrt{\pi}}. \quad (38)$$

Inserting (38) into (35) yields

$$4\sum_{l=0}^{z-1}\left(\begin{array}{c}z-1\\l\end{array}\right)\left(\frac{1}{2}\right)^{z-1}(l+1)\min\left(1,\frac{z/2}{l+1}\right)=2z+1-\frac{\sqrt{2}\sqrt{z}}{\sqrt{\pi}}. \quad (39)$$

By substituting (34) and (39) into (13") it is finally obtained

$$\left[\frac{d\pi(h_A^s)}{d\upsilon}\right]_{\upsilon_k=\upsilon=\frac{1}{2}} = -1 + \frac{2}{\sqrt{2}\sqrt{\pi}\sqrt{z}}. \tag{40}$$

The RHS of (40) is decreasing in z and converging to -1 as $z \to \infty$.

(iii) Inserting (34) and (40) into (11') gives:

$$\left[\frac{\partial \upsilon}{\partial p_A}\right]_{p_A=p_B\equiv p} = \left(1 - \frac{1}{\sqrt{2}\sqrt{\pi}\sqrt{z}}\right) \frac{1}{2\,(1-p)\left(\frac{2}{\sqrt{2}\sqrt{\pi}\sqrt{z}} - 1\right)}. \tag{41}$$

It is immediately seen that $[\partial\upsilon/\partial p_A]_{p_A=p_B} \to -\frac{1}{2(1-p)}$ as $z \to \infty$. Also:

$$\frac{\partial}{\partial z}\left[\frac{\partial \upsilon}{\partial p_A}\right]_{p_A=p_B\equiv p} = \frac{1}{1-p}\frac{1}{\sqrt{2}\sqrt{\pi}\sqrt{z^3}}\frac{1}{(\frac{4}{\sqrt{2}\sqrt{\pi}\sqrt{z}} - 2)^2} > 0.$$

■

Proof of part (ii) of Lemma 3. Unlike in the two-buyer case, proving concavity for general z is burdensome. To shorten notation, from now on we drop subscripts for variables at the simmetric MSE of the buyer game: we accordingly refer to $[\pi(h_A^s)]_{\upsilon_k=\upsilon}$, as $\pi(h_A^s)$, to $d\,[\pi(h_A^s)]_{\upsilon_k=\upsilon}/d\upsilon$ as $d\pi(h_A^s)/d\upsilon$, and so on. In (4'), Ey_A obviously decreases as p_A increases (and υ correspondingly decreases). Hence we are assured of concavity if

$$\frac{d}{dp_A}\left(p_A\frac{dEy_A}{d\upsilon}\frac{\partial\upsilon}{\partial p_A}\right) \le 0. \tag{42}$$

After rearrangement, this derivative can be written as

$$\frac{d}{dp_A}\left(p_A\frac{dEy_A}{d\upsilon}\frac{\partial\upsilon}{\partial p_A}\right) = \frac{dEy_A}{d\upsilon}\left(\frac{\partial\upsilon}{\partial p_A} + p_A\frac{\partial^2\upsilon}{\partial p_A^2}\right) + p_A\left[\frac{d^2Ey_A}{d\upsilon^2}\frac{\partial\upsilon}{\partial p_A} + \frac{dEy_A}{d\upsilon}\frac{\partial^2\upsilon}{\partial\upsilon\partial p_A}\right]\frac{\partial\upsilon}{\partial p_A}. \tag{42'}$$

We know that $dEy_A/d\upsilon \ge 0$[12]and $\partial\upsilon/\partial p_A < 0$. Note that $\partial^2\upsilon/\partial p_A^2 \le 0$ given that

$$\frac{\partial^2\upsilon}{\partial p_A^2} = \frac{1}{[\partial\varphi/\partial\upsilon]^2}\pi(h_A^s)\frac{d\pi(h_A^s)}{d\upsilon} \tag{43}$$

[12]It can be checked from (18) that $dEy_A/d\upsilon = 0$ at $\upsilon = 1$.

and $d\pi(h_A^s)/dv \leq 0$.[13] Hence, a sufficient condition for (42') to hold is

$$\frac{d^2 Ey_A}{dv^2}\frac{\partial v}{\partial p_A} + \frac{dEy_A}{dv}\frac{\partial^2 v}{\partial v \partial p_A} \geq 0, \tag{44}$$

that is,

$$\frac{d^2 Ey_A}{dv^2}\frac{\partial v}{\partial p_A} + \frac{dEy_A}{dv}\frac{1}{[\partial\varphi/\partial v]^2}\left(\frac{d\pi(h_A^s)}{dv}\frac{\partial\varphi}{\partial v} - \pi(h_A^s)\frac{\partial^2\varphi}{\partial v^2}\right) \geq 0. \tag{45}$$

Making use of (7), $\partial\varphi/\partial v$ and $\partial^2\varphi/\partial v^2$ can be written as

$$\frac{\partial\varphi}{\partial v} = (1-p_A)\left(\frac{d\pi(h_A^s)}{dv} - \frac{\pi(h_A^s)}{\pi(h_B^s)}\frac{d\pi(h_B^s)}{dv}\right), \tag{46}$$

and

$$\frac{\partial^2\varphi}{\partial v^2} = (1-p_A)\left(\frac{d^2\pi(h_A^s)}{dv^2} - \frac{\pi(h_A^s)}{\pi(h_B^s)}\frac{d^2\pi(h_B^s)}{dv^2}\right). \tag{47}$$

By inserting (46) and (47) into (45), this becomes

$$\begin{aligned}
&\frac{dEy_A}{dv}\frac{d\pi(h_A^s)}{dv}\left(\frac{d\pi(h_A^s)}{dv}\pi(h_B^s) - \frac{d\pi(h_B^s)}{dv}\pi(h_A^s)\right)\\
&+\pi(h_A^s)\left[\frac{d^2 Ey_A}{dv^2}\left(\frac{d\pi(h_A^s)}{dv}\pi(h_B^s) - \frac{d\pi(h_B^s)}{dv}\pi(h_A^s)\right)\right.\\
&\left.-\frac{dEy_A}{dv}\left(\frac{d^2\pi(h_A^s)}{dv^2}\pi(h_B^s) - \frac{d^2\pi(h_B^s)}{dv^2}\pi(h_A^s)\right)\right] \geq 0.
\end{aligned} \tag{48}$$

Since the expression on the first line is always nonnegative, a sufficient condition for (48) to hold is

$$\begin{aligned}
&\frac{d^2 Ey_A}{dv^2}\left(\frac{d\pi(h_A^s)}{dv}\pi(h_B^s) - \frac{d\pi(h_B^s)}{\partial v}\pi(h_A^s)\right)\\
&-\frac{dEy_A}{dv}\left(\frac{d^2\pi(h_A^s)}{dv^2}\pi(h_B^s) - \frac{d^2\pi(h_B^s)}{dv^2}\pi(h_A^s)\right) \geq 0
\end{aligned} \tag{49}$$

Validity of (49) follows from the fact that both of the two following inequalities hold:

$$\frac{d^2 Ey_A}{dv^2}\frac{d\pi(h_A^s)}{dv} - \frac{dEy_A}{dv}\frac{d^2\pi(h_A^s)}{dv^2} \geq 0, \tag{50}$$

[13] It can be checked from (9) that $d\pi(h_A^s)/dv = 0$ at $v = 0$.

$$\frac{d^2 Ey_A}{dv^2}\frac{d\pi(h_B^s)}{dv} - \frac{dEy_A}{dv}\frac{d^2\pi(h_B^s)}{dv^2} \leq 0. \tag{51}$$

The argument establishing either inequality is long and tedious. In what follows we will establish (51) (the argument would run along similar lines for (50)). The uninterested reader might skip the proof and nevertheless be persuaded of (50) and (51) by running simulations through a package such as Maple: it would be found that, no matter the value of z being tried, (50) and (51) are met over the entire interval $[0, 1]$ of v.

The proof exploits a few additional relationships between variables of concern. By comparing (5) with (17) it is readily found that[14]

$$Ey_A = zv\pi(h_A^s). \tag{52}$$

Consequently,

$$\frac{dEy_A}{dv} = z\left(\pi(h_A^s) + v\frac{d\pi(h_A^s)}{dv}\right) \tag{53}$$

and

$$\frac{d^2 Ey_A}{dv^2} = z\left(2\frac{d\pi(h_A^s)}{dv} + v\frac{d^2\pi(h_A^s)}{dv^2}\right). \tag{54}$$

It is also helpful to relate $d^2\pi(h_A^s)/dv^2$, $d\pi(h_A^s)/dv$, and $\pi(h_A^s)$ to each other. Note that, for any $l \in \{2, ..., z-1\}$ and $l' = l - 1$,

$$l\begin{pmatrix} z-1 \\ l \end{pmatrix} v^{l-1}(1-v)^{z-1-l} = (z-1-l')\begin{pmatrix} z-1 \\ l' \end{pmatrix} v^{l'}(1-v)^{z-2-l'}.$$

In view of this (9) can be rewritten as

$$\frac{d\pi(h_A^s)}{dv} = -\frac{z}{2}\sum_{l=\frac{z}{2}}^{z-1}\begin{pmatrix} z-1 \\ l \end{pmatrix} v^{l-1}(1-v)^{z-1-l}\frac{1}{l+1}. \tag{9'}$$

By comparing (9') with (5) it is seen that

$$\frac{d\pi(h_A^s)}{dv} = -\frac{1}{v}\left(\pi(h_A^s) - \sum_{l=0}^{\frac{z}{2}-1}\begin{pmatrix} z-1 \\ l \end{pmatrix} v^{l}(1-v)^{z-1-l}\right). \tag{55}$$

[14] This follows from the fact that, for any $l = 1, .., z$,

$$\begin{pmatrix} z \\ l \end{pmatrix} v^l(1-v)^{z-l} = zv\begin{pmatrix} z-1 \\ l-1 \end{pmatrix}\frac{z}{l+1}v^{l-1}(1-v)^{z-1-(l-1)}.$$

Differentiation of (55) then leads to

$$\frac{d^2\pi(h_A^s)}{dv^2} = -\frac{2}{v}\frac{d\pi(h_A^s)}{dv} - \left(\begin{array}{c} z-1 \\ \frac{z}{2} \end{array}\right) v^{\frac{z}{2}-2}(1-v)^{\frac{z}{2}-1}\frac{z}{2}. \tag{56}$$

We proceed likewise with regard to $d^2\pi(h_B^s)/dv^2$, $d\pi(h_B^s)/dv$, and $\pi(h_B^s)$. First, (10) can be rewritten as

$$\frac{d\pi(h_B^s)}{dv} = \frac{z}{2}\sum_{l=\frac{z}{2}}^{z-1}\left(\begin{array}{c} z-1 \\ l \end{array}\right)(1-v)^{l-1}v^{z-1-l}\frac{1}{l+1}. \tag{10'}$$

Second, from a comparison of (10') and (6) it is found that

$$\frac{d\pi(h_B^s)}{dv} = \frac{1}{1-v}\left(\pi(h_B^s) - \sum_{l=0}^{\frac{z}{2}-1}\left(\begin{array}{c} z-1 \\ l \end{array}\right)(1-v)^l v^{z-1-l}\right). \tag{57}$$

Finally, differentiation of (57) yields

$$\frac{d^2\pi(h_B^s)}{dv^2} = \frac{2}{1-v}\frac{d\pi(h_B^s)}{dv} - \left(\begin{array}{c} z-1 \\ \frac{z}{2} \end{array}\right)(1-v)^{\frac{z}{2}-2}v^{\frac{z}{2}-1}\frac{z}{2}. \tag{58}$$

Substituting all the above equations into (51) and dividing by $-(1-v)^{z-2}$, it is finally obtained

$$\begin{aligned} &\frac{z}{2}\sum_{l=\frac{z}{2}}^{z-1}\left(\begin{array}{c} z-1 \\ l \end{array}\right)(1-v)^{l-\frac{z}{2}}v^{z-1-l}\frac{1}{l+1}\left[\left(\begin{array}{c} z-1 \\ \frac{z}{2} \end{array}\right)v^{\frac{z}{2}-1}\frac{z}{2}\right. \\ &\qquad\left. +2\sum_{l=0}^{\frac{z}{2}-1}\left(\begin{array}{c} z-1 \\ l \end{array}\right)v^l(1-v)^{\frac{z}{2}-1-l}\right] \\ &-\frac{z}{2}\left(\begin{array}{c} z-1 \\ \frac{z}{2} \end{array}\right)v^{\frac{z}{2}-1}\sum_{l=0}^{\frac{z}{2}-1}\left(\begin{array}{c} z-1 \\ l \end{array}\right)v^l(1-v)^{\frac{z}{2}-1-l} \geq 0. \end{aligned} \tag{59}$$

The negative part in the LHS of (59) is made up of the expression on the last line. By letting $h = \frac{z}{2}+1-l$, this is written as

$$-\frac{z}{2}\left(\begin{array}{c} z-1 \\ \frac{z}{2} \end{array}\right)\sum_{h=2}^{\frac{z}{2}+1}\left(\begin{array}{c} z-1 \\ \frac{z}{2}+1-h \end{array}\right)v^{z-h}(1-v)^{h-2}.$$

On close scrutiny, in the LHS of (59) also the positive part contains terms in $\upsilon^{z-h}(1-\upsilon)^{h-2}$ (along with other terms): each such term is equal to

$$\frac{z}{2}\left[\sum_{j=1}^{h-1}\binom{z-1}{\frac{z}{2}-1+j}\binom{z-1}{\frac{z}{2}+j-h}\frac{4}{z+2j} + \binom{z-1}{\frac{z}{2}}\binom{z-1}{\frac{z}{2}-2+h}\frac{z}{z+2h-2}\right]\upsilon^{z-h}(1-\upsilon)^{h-2}.$$

Validity of (59) can thus be established by showing that

$$-\binom{z-1}{\frac{z}{2}}\binom{z-1}{\frac{z}{2}+1-h}+\sum_{j=1}^{h-1}\binom{z-1}{\frac{z}{2}-1+j}\binom{z-1}{\frac{z}{2}+j-h}\frac{4}{z+2j} + \binom{z-1}{\frac{z}{2}}\binom{z-1}{\frac{z}{2}-2+h}\frac{z}{z+2h-2}\geq 0 \quad \forall h=2,...,\frac{z}{2}+1. \tag{60}$$

For $h \in \{2,3\}$ validity of (60) can be checked by substitution. As for any $3 < h \leq \frac{z}{2}+1$, note that, in the sum on the first line of (60), any j term has the same binomial coefficient as any $j' = (h-j)$ one. Thus we have

$$\sum_{j=1}^{h-1}\binom{z-1}{\frac{z}{2}-1+j}\binom{z-1}{\frac{z}{2}+j-h}\frac{4}{z+2j} = \sum_{j=1}^{\frac{h-1}{2}}\binom{z-1}{\frac{z}{2}-1+j}\binom{z-1}{\frac{z}{2}+j-h}\frac{8z+8h}{(z+2j)(z+2h-2j)} \quad \forall h\in\left\{5,7,...\frac{z}{2}+1\right\} \tag{61}$$

and

$$\sum_{j=1}^{h-1}\binom{z-1}{\frac{z}{2}-1+j}\binom{z-1}{\frac{z}{2}+j-h}\frac{4}{z+2j} = \sum_{j=1}^{\frac{h-2}{2}}\binom{z-1}{\frac{z}{2}-1+j}\binom{z-1}{\frac{z}{2}+j-h}\frac{8z+8h}{(z+2j)(z+2h-2j)} + \binom{z-1}{\frac{z}{2}-1+\frac{h}{2}}\binom{z-1}{\frac{z}{2}-\frac{h}{2}}\frac{4}{z+h} \quad \forall h\in\left\{4,6,...,\frac{z}{2}\right\}. \tag{62}$$

We first deal with the case of any odd h. Inserting (61) into (60) leads to

$$\binom{z-1}{\frac{z}{2}}\binom{z-1}{\frac{z}{2}+1-h}\frac{(5-h)2z+4h+4}{(z+2)(z+2h-2)}$$
$$+\sum_{j=2}^{\frac{h-1}{2}}\binom{z-1}{\frac{z}{2}-1+j}\binom{z-1}{\frac{z}{2}+j-h}\frac{8z+8h}{(z+2j)(z+2h-2j)}\geq 0. \qquad (60')$$

It is important to note that, for any $j=2,...,\frac{h-1}{2}$,

$$\binom{z-1}{\frac{z}{2}-1+j}\binom{z-1}{\frac{z}{2}+j-h}=$$
$$\binom{z-1}{\frac{z}{2}}\binom{z-1}{\frac{z}{2}+1-h}\Pi_{k=1}^{j-1}\frac{z-2k}{z+2k}\Pi_{r=2}^{j}\frac{z+2h-2r}{z-2h+2r}. \qquad (63)$$

By taking (63) into account and simplifying, (60') becomes

$$\binom{z-1}{\frac{z}{2}}\binom{z-1}{\frac{z}{2}+1-h}\left[\frac{(5-h)2z+4h+4}{(z+2)(z+2h-2)}\right.$$
$$+(8z+8h)\left(\frac{z-2}{(z+2)(z+4)(z-2h+4)}\right.$$
$$\left.\left.+\sum_{j=3}^{\frac{h-1}{2}}\frac{\Pi_{k=1}^{j-1}(z-2k)}{\Pi_{k=1}^{j}(z+2k)}\frac{\Pi_{r=2}^{j-1}(z+2h-2r)}{\Pi_{r=2}^{j}(z-2h+2r)}\right)\right]\geq 0. \qquad (64)$$

Validity of (64) is immediate for $h=5$. For any $h\in\{7,9,...\}$, it will be proved by establishing

$$-2h\frac{z}{(z+2)(z+2h-2)}+8\left(\frac{z(z-2)}{(z+2)(z+4)(z-2h+4)}\right.$$
$$\left.+z\sum_{j=3}^{\frac{h-1}{2}}\frac{\Pi_{k=1}^{j-1}(z-2k)}{\Pi_{k=1}^{j}(z+2k)}\frac{\Pi_{r=2}^{j-1}(z+2h-2r)}{\Pi_{r=2}^{j}(z-2h+2r)}\right)\geq 0, \qquad (65)$$

or, more succintly,

$$-2hf_1+8\left(f_2+\sum_{j=3}^{\frac{h-1}{2}}f_j\right)\geq 0, \qquad (66)$$

where

$$f_1 = \frac{z}{(z+2)(z+2h-2)}$$

$$f_2 = \frac{z(z-2)}{(z+2)(z+4)(z-2h+4)}$$

$$f_j = \frac{z\Pi_{k=1}^{j-1}(z-2k)\,\Pi_{r=2}^{j-1}(z+2h-2r)}{\Pi_{k=1}^{j}(z+2k)\Pi_{r=2}^{j}(z-2h+2r)}.$$

It is readily seen that $f_2 > f_1$ for any $h \in \{7, 9, ...\frac{z}{2}+1\}$ and that $f_{j+1} > f_j$ for any $j = 2, ..., \frac{h-1}{2} - 1$. Consequently $-2hf_1 + 8\left(f_2 + \sum_{j=3}^{\frac{h-1}{2}} f_j\right) > \left(-2h + 8\left(1 + \frac{h-1}{2} - 2\right)\right) f_1$. Thus we only need to show that $-2h + 8\left(\frac{h-1}{2} - 1\right) \geq 0$ which is actually the case for any $h > 5$.

The argument runs along similar lines for h an even number. Inserting (62) into (60) and using (63), we now get:

$$\begin{aligned}
&\binom{z-1}{\frac{z}{2}}\binom{z-1}{\frac{z}{2}+1-h}\Bigg[\frac{(5-h)2z+4h+4}{(z+2)(z+2h-2)} \\
&+(8z+8h)\Bigg(\frac{z-2}{(z+2)(z+4)(z-2h+4)} \\
&+\sum_{j=3}^{\frac{h}{2}-1}\frac{\Pi_{k=1}^{j-1}(z-2k)}{\Pi_{k=1}^{j}(z+2k)}\frac{\Pi_{r=2}^{j-1}(z+2h-2r)}{\Pi_{r=2}^{j}(z-2h+2r)}\Bigg) \\
&+\frac{\Pi_{k=1}^{\frac{h}{2}-1}(z-2k)}{\Pi_{k=1}^{\frac{h}{2}-1}(z+2k)}\frac{\Pi_{r=2}^{\frac{h}{2}-1}(z+2h-2r)}{\Pi_{r=2}^{\frac{h}{2}}(z-2h+2r)}4\Bigg] \geq 0. \qquad (60")
\end{aligned}$$

A sufficient condition for (60") to hold is then

$$\begin{aligned}
&\frac{-2h}{(z+2)(z+2h-2)}z + 8\Bigg(\frac{z(z-2)}{(z+2)(z-2h+4)(z+4)} \\
&+z\sum_{j=3}^{\frac{h}{2}-1}\frac{\Pi_{k=1}^{j-1}(z-2k)}{\Pi_{k=1}^{j}(z+2k)}\frac{\Pi_{r=2}^{j-1}(z+2h-2r)}{\Pi_{r=2}^{j}(z-2h+2r)}\Bigg) \\
&+4\frac{\Pi_{k=1}^{\frac{h}{2}-1}(z-2k)}{\Pi_{k=1}^{\frac{h}{2}-1}(z+2k)}\frac{\Pi_{r=2}^{\frac{h}{2}-1}(z+2h-2r)}{\Pi_{r=2}^{\frac{h}{2}}(z-2h+2r)} \geq 0, \qquad (67)
\end{aligned}$$

or

$$-2hf_1 + 8\left(f_2 + \sum_{j=3}^{\frac{h}{2}-1} f_j\right) + 4\frac{z+h}{z} f_{j=\frac{h}{2}} \geq 0. \tag{68}$$

Arguing as above, it suffices that $-2h + 8(\frac{h}{2} - 2) + 4 \geq 0$, which is actually the case for any $h \geq 6$. ■

z	$[\pi(h^s)]_{v_h=1/2}$	p^*	p^{**}			
			$T=10$	$T=20$	$T=40$	$T=80$
2	.75	.5	.934	.965	.9819	.990
4	.81	.55	.943	.969	.9843	.9920
10	.87	.60	.949	.972	.9859	.9928
20	.91	.62	.951	.974	.9866	.9932
50	.94	.639	.953	.9752	.9871	.9934
100	.96	.647	.954	.9757	.9874	.9936
400	.98	.657	.955	.9762	.9876	.9937
$5,000$	.99	.66	.956	.9766	.9878	.9938

TABLE I